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On Abel–Tauber Theorems for Fourier Cosine Transforms

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We prove Abel–Tauber theorems for Fourier cosine transforms. They treat the boundary cases of the Abel–Tauber theorem of Pitman, and Soni and Soni. A similar result for Fourier cosine series is obtained as a corollary. The latter gives an answer to an open problem in Boas' book on Fourier series. Applications to probability distributions and stationary processes are given.

1. INTRODUCTION AND RESULTS

The aim of this paper is to prove Abel–Tauber theorems for Fourier cosine series and integrals. For example, we characterize the asymptotic behavior $f(t) \sim t^{-1}$ as $t \to \infty$ in terms of the Fourier cosine transform of f, where f is a locally integrable, eventually non-increasing function on $[0, \infty)$ such that $\lim_{t\to\infty} f(t) = 0$.

To state our results, we recall and introduce some notation. We denote by R_0 the whole class of slowly varying functions at infinity; that is, R_0 is the class of positive measurable l, defined on some neighborhood of infinity, satisfying

$$\forall \lambda > 0, \qquad \lim_{x\to\infty} l(\lambda x)/l(x) = 1.$$

For $l \in R_0$, the class Π_l is the class of measurable g, defined on some neighborhood of infinity, satisfying

$$\forall \lambda \geq 1, \qquad \lim_{x\to\infty} \{g(\lambda x) - g(x)\}/l(x) = c \log \lambda$$

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for some constant c called the *l-index* of g. It is useful to name the class of functions of which we define the Fourier cosine transforms. The function $f\colon [0, \infty) \to \mathbb{R}$ belongs to $D^1_{\mathrm{loc}}[0, \infty)$ if it is locally integrable and eventually non-increasing on $[0, \infty)$, $\lim_{t\to\infty} f(t) = 0$. For $f \in D^1_{\mathrm{loc}}[0, \infty)$, we define the *Fourier cosine transform* F_c of f by

$$F_c(\xi) = \int_0^{\infty-} f(t)\cos t\xi\, dt \qquad (0 < \xi < \infty), \tag{1.1}$$

where we write $\int_a^{\infty-}$ to denote an improper integral obtained from $\int_a^M$ by letting $M \uparrow \infty$. Since the improper integral on the right converges uniformly on each (ε, ∞) with $\varepsilon > 0$, F_c is a continuous function on $(0, \infty)$. See the proof of Theorem 6 of Titchmarsh [7].

Here are the main theorems of this paper:

THEOREM 1.1. *Let $l \in R_0$ and $f \in D^1_{\mathrm{loc}}[0, \infty)$. Let F_c be the Fourier cosine transform of f. Then the following are equivalent:*

$$f(t) \sim t^{-1}l(t) \qquad (t \to \infty), \tag{1.2}$$

$$F_c(1/\cdot) \in \Pi_l \textit{ with l-index } 1. \tag{1.3}$$

THEOREM 1.2. *Let $l \in R_0$ and $f \in D^1_{\mathrm{loc}}[0, \infty)$. Let F_c be the Fourier cosine transform of f. Then*

$$F_c(\xi) \sim \xi^{-1}l(1/\xi)\frac{\pi}{2} \qquad (\xi \to 0+) \tag{1.4}$$

implies

$$f \in \Pi_l \textit{ with l-index } -1. \tag{1.5}$$

Conversely, if F_c is non-negative and non-increasing in a neighborhood of zero, then (1.5) *implies* (1.4).

We note that in Theorem 1.1 the Abelian implication (1.2) $\Rightarrow$ (1.3) is essentially due to Pitman [5].

The analogue of Theorem 1.1 for Fourier cosine series is:

THEOREM 1.3. *Let $l \in R_0$. Suppose that the real sequence $\{a_n\}$ is eventually non-increasing, and tends to 0 as $n \to \infty$. We set*

$$\hat{a}_c(\xi) = \sum_{n=0}^{\infty} a_n \cos n\xi \qquad (0 < \xi < 2\pi). \tag{1.6}$$

Then the following are equivalent:

$$a_n \sim n^{-1}l(n) \qquad (n \to \infty), \tag{1.7}$$

$$\hat{a}_c(1/\cdot) \in \Pi_l \textit{ with l-index } 1. \tag{1.8}$$

Let K be a positive constant. If we set $l(t) \equiv K$ in Theorem 1.3, then we obtain an answer to an open problem in Boas [3, p. 45].

Now we recall the Abel–Tauber theorem of Pitman [5], and Soni and Soni [6], which is closely related to Theorems 1.1 and 1.2.

THEOREM 1.4 [5, 6]. *Let $0 < \alpha < 1$, $l \in R_0$ and $f \in D^1_{\text{loc}}[0, \infty)$. Let F_c be the Fourier cosine transform of f. Then the following are equivalent:*

$$f(t) \sim t^{-\alpha}l(t) \qquad (t \to \infty), \tag{1.9}$$

$$F_c(\xi) \sim \xi^{-(1-\alpha)}l(1/\xi)\Gamma(1-\alpha)\sin(\pi\alpha/2) \qquad (\xi \to 0+). \tag{1.10}$$

We consider Theorem 1.1 (Theorem 1.2 resp.) to be an analogue of Theorem 1.4 for the boundary case $\alpha = 1$ ($\alpha = 0$ resp.). From Theorems 1.1 and 1.2, we see that the cases $\alpha = 0, 1$ are critical ones for Fourier cosine transforms, and that they require Π-variation for their characterizations.

For absolutely convergent integral transforms like Laplace transforms, there already exist Abel–Tauber theorems which involve Π-variation. See de Haan [4], and Bingham and Teugels [1] as well as Bingham *et al.* [2, Chap. 4]. Our Theorems 1.1 and 1.2 are different from the previous works in that they involve both Π-variation and improper integrals.

We have the inversion formula for Fourier cosine transforms (see Theorem 6 of [7]) but the integral which appears in the formula is improper. So it is difficult to make direct use of it to prove the Tauberian implications such as (1.3) $\Rightarrow$ (1.2), and (1.4) $\Rightarrow$ (1.5). The key to the proofs is to reduce the problem to a completely monotone f. We use this method in the proofs of Theorems 1.1 and 1.2. We will prove Theorem 1.3 as a corollary of Theorem 1.1.

The plan of this paper is as follows: In Section 2, we prove Theorems 1.1 and 1.3. In Section 3, we prove Theorem 1.2. In Section 4, we apply Theorems 1.1 and 1.2 to the tail behavior of a probability distribution. Finally in Section 5, we apply Theorems 1.1 and 1.2 to stationary processes.

2. PROOFS OF THEOREMS 1.1 AND 1.3

Proof of Theorem 1.1. *Step* 1. Choose $M > 0$ so large that f is nonincreasing on $[M, \infty)$. Set

$$f^M(t) = \begin{cases} f(M) & (0 \le t < M), \\ f(t) & (M \le t < \infty). \end{cases}$$

Let F_c^M be the Fourier cosine transform of f^M:

$$F_c^M(\xi) = \int_0^{\infty -} f^M(t)\cos t\xi \,\mathrm{dt} \qquad (\xi > 0).$$

Then, for any $\lambda > 1$,

$$\begin{aligned}
&|\{F_c^M(1/\lambda x) - F_c^M(1/x)\} - \{F_c(1/\lambda x) - F_c(1/x)\}|/l(x) \\
&= \frac{1}{l(x)}\left|\int_0^M \{f(M) - f(t)\}\{\cos(t/\lambda x) - \cos(t/x)\}\,dt\right| \\
&\leq \frac{(1-\lambda^{-1})}{xl(x)}\int_0^M t|f(M) - f(t)|dt \to 0 \qquad (x \to \infty),
\end{aligned}$$

whence (1.3) holds if and only if $F_c^M(1/\cdot) \in \Pi_l$ with l-index 1. Thus we may assume that f is finite and non-increasing on $[0, \infty)$.

First we prove the Abelian implication (1.2) $\Rightarrow$ (1.3). By [5, Theorem 7(iii)],

$$F_c(1/x) - \int_0^x f(t)\,dt \sim -\gamma l(x) \qquad (x \to \infty),$$

where γ is Euler's constant. By [2, Theorem 3.6.8], $\int_0^x f(t)\,dt \in \Pi_l$ in x with l-index 1. So for any $\lambda \geq 1$,

$$\begin{aligned}
\frac{F_c(1/\lambda x) - F_c(1/x)}{l(x)} &= \frac{F_c(1/\lambda x) - \int_0^{\lambda x} f(t)\,dt}{l(x)} \\
&\quad + \frac{\int_0^{\lambda x} f(t)\,dt - \int_0^x f(t)\,dt}{\pi l(x)} - \frac{F_c(1/x) - \int_0^x f(t)\,dt}{l(x)} \\
&\to \log \lambda \qquad (x \to \infty),
\end{aligned}$$

whence (1.3).

Step 2. By the second mean-value theorem for integrals, $\xi F_c(\xi)$ is bounded on $(0, \infty)$ (see [2, p. 241]). By [2, Theorem 3.7.4], (1.3) implies $|F_c(1/\cdot)| \in R_0$, and so $F_c(\cdot) \in L^1_{\mathrm{loc}}[0, \infty)$. Then [7, Theorem 38] gives

$$t^{-1/2}\int_0^t f(u)\,du = \frac{2}{\pi}\int_0^\infty \frac{F_c(\xi)}{\xi^{1/2}} \cdot \frac{\sin t\xi}{(t\xi)^{1/2}}\,d\xi \qquad (t > 0). \tag{2.1}$$

See also [2, p. 240]. Since $t^{-1/2}\sin t$ is bounded on $(0, \infty)$ and $\xi^{-1/2}F_c(\xi)$ is integrable over $(0, \infty)$, we have dominated convergence, as $t \to \infty$, in (2.1), and so

$$\lim_{t\to\infty} t^{-1/2}\int_0^t f(u)\,du = 0.$$

Therefore, integrating by parts,

$$\int_1^\infty f(t)\,dt/t = \int_1^\infty \left(t^{-1/2}\int_1^t f(u)\,du\right) t^{-3/2}\,dt < \infty.$$

Step 3. We define a measure σ on $(0, \infty)$ by

$$\sigma(d\lambda) = I_{(0,1)}(\lambda) f(1/\lambda)\,d\lambda/\lambda.$$

Then σ is finite because

$$\sigma(0, \infty) = \int_0^1 f(1/\lambda)\,d\lambda/\lambda = \int_1^\infty f(t)\,dt/t < \infty.$$

We set

$$g(t) = \int_0^\infty e^{-t\lambda}\sigma(d\lambda) \qquad (t \geq 0). \tag{2.2}$$

Then g is finite and non-increasing on $[0, \infty)$, $\lim_{t\to\infty} g(t) = 0$. Since $\log\{xf(x)\}$ is slowly increasing, by Karamata's theorem, (1.2) is equivalent to

$$g(t) \sim t^{-1}l(t) \qquad (t \to \infty). \tag{2.3}$$

Let G_c be the Fourier cosine transform of g:

$$G_c(\xi) = \int_0^{\infty-} g(t)\cos t\xi\,dt \qquad (\xi > 0).$$

Then for $\xi > 0$,

$$G_c(\xi) = \int_0^\infty \frac{\lambda}{\lambda^2 + \xi^2}\sigma(d\lambda) = \int_0^\infty \frac{f(t)}{1 + t^2\xi^2}\,dt - \int_0^1 \frac{f(t)}{1 + t^2\xi^2}\,dt.$$

Since

$$\frac{1}{1+t^2\xi^2} = \int_0^\infty \xi^{-1} e^{-u/\xi} \cos ut\, du \qquad (t>0, \xi>0), \tag{2.4}$$

[7, Theorem 36] yields

$$\int_0^\infty \frac{f(t)}{1+t^2\xi^2}\, dt = \int_0^\infty \xi^{-1} e^{-u/\xi} F_c(u)\, du \qquad (\xi > 0).$$

Hence, for any $\lambda > 1$ and $x > 0$,

$$\begin{aligned} \frac{G_c(1/\lambda x) - G_c(1/x)}{l(x)} &= \int_0^\infty \frac{F_c(u/\lambda x) - F_c(u/x)}{l(x)} e^{-u}\, du \\ &\quad - \frac{(1-\lambda^{-2})}{x^2 l(x)} \int_0^1 \frac{t^2 f(t)}{\{1+(t/x)^2\}\{1+(t/\lambda x)^2\}}\, dt. \end{aligned} \tag{2.5}$$

The second term on the right clearly tends to zero as $x \to \infty$. Now since $F_c(\xi) \to 0$ as $\xi \to \infty$, $F_c(1/\cdot)$ can be extended to a continuous function on $[0, \infty)$. Therefore by [2, Theorem 3.8.6] we have dominated convergence, as $x \to \infty$, in the first term on the right, so it converges to $\log \lambda$. Thus (1.3) implies

$$G_c(1/\cdot) \in \Pi_l \text{ with } l\text{-index } 1. \tag{2.6}$$

Therefore, in order to prove (1.3) $\Rightarrow$ (1.2), it is enough to prove (2.6) $\Rightarrow$ (2.3).

Step 4. By [7, Theorem 6],

$$g(t) = \frac{2}{\pi} \int_0^{\infty -} G_c(\xi) \cos t\xi\, d\xi \qquad (t > 0).$$

Since $\xi G_c(\xi)$ is bounded on $(0, \infty)$,

$$\lim_{\xi \to \infty} G_c(\xi) \sin t\xi = 0,$$

while, by (2.6) and [2, Theorem 3.7.4],

$$\lim_{\xi \to 0+} G_c(\xi) \sin t\xi = 0.$$

So, integrating by parts,

$$tg(t) = \int_{0+}^{\infty-} k(u)L(tu)\,du \qquad (t > 0), \tag{2.7}$$

where $\int_{0+}^{a} = \lim_{\varepsilon \downarrow 0} \int_{\varepsilon}^{a}$, and

$$k(t) = \frac{2}{\pi} \cdot \frac{\sin(1/t)}{t} \qquad (t > 0),$$

$$L(t) = t\{G_c(1/t)\}' \qquad (t > 0).$$

From

$$G_c(1/t) = \int_0^{\infty} \frac{\lambda}{(\lambda^2 + t^{-2})}\,\sigma(d\lambda) \qquad (t > 0),$$

we get

$$\{G_c(1/t)\}' = \frac{2}{t^3} \int_0^{\infty} \frac{\lambda}{(\lambda^2 + t^{-2})^2}\,\sigma(d\lambda) \qquad (t > 0).$$

So $\log\{G_c(1/t)\}'$ is slowly decreasing, whence, by [2, Theorem 3.6.10], (2.6) implies $L(t) \sim l(t)$ as $t \to \infty$.

Now we have

$$L(t) = 2t^2 \int_0^{\infty} \frac{\lambda}{(1 + t^2\lambda^2)^2}\,\sigma(d\lambda) \qquad (t > 0).$$

We set $L(0) = 0$. Then we easily see that $L \in C^1[0, \infty)$; in particular, L is locally of bounded variation on $[0, \infty)$. We define two positive, non-decreasing functions ϕ_1, ϕ_2 by

$$\phi_1(t) = 2t^2 \qquad (t > 0),$$

$$\phi_2(t) = 1 \Big/ \left\{ \int_0^{\infty} \frac{\lambda}{(1 + t^2\lambda^2)^2}\,\sigma(d\lambda) \right\} \qquad (t > 0).$$

Then we have

$$\phi_i(2t) = O(\phi_i(t)) \qquad (i = 1, 2)\ (t \to \infty),$$

$$L(t) = \phi_1(t)/\phi_2(t) \qquad (t > 0).$$

Therefore, by the Quasi-Monotonicity Theorem (see [2, Theorem 2.7.2]), L is quasi-monotone; that is, for some $\delta > 0$,

$$\int_0^t u^\delta |dL(u)| = O(t^\delta L(t)) \qquad (t \to \infty),$$

where $|dL(u)|$ is the variation measure of L. Since

$$\int_t^\infty k(u)\, du = O(1/t) \qquad (t \to \infty),$$

$$\int_{0+}^t k(u)\, du = O(t) \qquad (t \to 0+),$$

by applying the theorem of Bojanic and Karamata (see [2, Theorem 4.1.5]) to (2.7) we obtain

$$tg(t) \sim l(t) \qquad (t \to \infty),$$

whence (2.3). This completes the proof. ∎

Proof of Theorem 1.3. We set

$$f(t) = \begin{cases} 2a_0 & (0 \le t \le \frac{1}{2}), \\ a_n & (n - \frac{1}{2} < t \le n + \frac{1}{2}, n = 1, 2, \ldots). \end{cases}$$

Let F_c be the Fourier cosine transform of f. Then

$$F_c(\xi) = \frac{\sin(\xi/2)}{(\xi/2)} \hat{a}_c(\xi) \qquad (0 < \xi < 2\pi). \tag{2.8}$$

By (2.8) and [2, Theorem 3.7.4], we easily see that (1.8) holds if and only if $F_c(1/\cdot) \in \Pi_l$ with l-index 1. Therefore, by Theorem 1.1, we obtain the theorem. ∎

3. Proof of Theorem 1.2

To prove Theorem 1.2, we need the following version of the Abel–Tauber theorem of de Haan [4].

THEOREM 3.1 (A Version of de Haan's Abel–Tauber Theorem). *Let $l \in R_0$ and $c \ge 0$. Let U be a non-decreasing, right-continuous function on $[0, \infty)$. Assume its Laplace–Stieltjes transform $\hat{U}(s) := \int_{[0,\infty)} e^{-s\lambda}\, dU(\lambda)$ is finite for any $s > 0$. Then the following are equivalent:*

(1) $U(1/\cdot) \in \Pi_l$ *with l-index $-c$,*

(2) $\hat{U} \in \Pi_l$ *with l-index $-c$.*

The proof of Theorem 3.1 is almost the same as that of [2, Theorem 3.9.1] except for that we use [2, Theorem 3.7.1(iii)] instead of [2, Theorem 3.7.1(ii)].

Proof of Theorem 1.2. In the same way as the proof of Theorem 1.1, we may assume that f is finite, positive, and non-increasing on $[0, \infty)$. We may also assume that f is left-continuous. We set $U(\lambda) = f(1/\lambda)$ for $\lambda > 0$, and $U(0) = 0$. Then U is a finite, non-decreasing, and right-continuous function on $[0, \infty)$. We define a finite measure σ on $(0, \infty)$ by

$$\sigma(d\lambda) = dU(\lambda).$$

We define g by (2.2). Then, by Theorem 3.1, (1.5) holds if and only if

$$g \in \Pi_l \text{ with } l\text{-index } -1. \tag{3.1}$$

Let G_c be the Fourier cosine transform of g. Then integration by parts yields, for any $\xi > 0$,

$$G_c(\xi) = \int_0^\infty \frac{\lambda}{\lambda^2 + \xi^2}\,\sigma(d\lambda) = \int_0^\infty \frac{1 - t^2\xi^2}{(1 + t^2\xi^2)^2}\, f(t)\, dt.$$

By (2.4),

$$\frac{1 - t^2\xi^2}{(1 + t^2\xi^2)^2} = \frac{\partial}{\partial t}\left(\frac{t}{1 + t^2\xi^2}\right) = \int_0^\infty \xi^{-2} u e^{-u/\xi} \cos ut\, du \qquad (t > 0, \xi > 0),$$

whence [7, Theorem 36] gives

$$G_c(\xi) = \xi^{-2} \int_0^\infty F_c(u) u e^{-u/\xi}\, du \qquad (\xi > 0). \tag{3.2}$$

Since $uF_c(u)$ is bounded on $(0, \infty)$, (1.4) implies

$$G_c(\xi) \sim \xi^{-1} l(1/\xi) \frac{\pi}{2} \qquad (\xi \to 0+). \tag{3.3}$$

Therefore, in order to prove (1.4) $\Rightarrow$ (1.5), it is enough to prove (3.3) $\Rightarrow$ (3.1).

By the representation

$$-\dot{g}(t) = \int_0^\infty e^{-t\lambda} \lambda \sigma(d\lambda) \qquad (t > 0),$$

we see that $-\dot{g}$ is locally integrable and non-increasing on $[0, \infty)$, $\lim_{t\to\infty}\dot{g}(t) = 0$. Therefore, by de Haan's theorem (see [2, Theorem 3.6.8]), (3.1) is equivalent to

$$-\dot{g}(t) \sim t^{-1}l(t) \qquad (t \to \infty). \tag{3.4}$$

Now integration by parts yields

$$\xi G_c(\xi) = \int_0^{\infty -} \{-\dot{g}(t)\}\sin t\xi \, dt \qquad (\xi > 0).$$

So, by [5, Theorems 1 and 5], (3.3) and (3.4) are equivalent. Thus (1.4) $\Rightarrow$ (1.5).

Finally, if F_c is non-negative and non-increasing in a neighborhood of 0, then, by Karamata's Tauberian theorem, (3.3) conversely implies (1.4), whence (1.5) $\Rightarrow$ (1.4). This completes the proof. ∎

4. Application to Probability Distributions

Let X be a real random variable defined on a probability space $(\Omega, \mathcal{F}, P)$. We define the *tail difference* D of X by

$$D(x) = P(X > x) - P(X \leq -x) \qquad (x \geq 0).$$

Let V be the imaginary part of the characteristic function of X:

$$V(\xi) = E[\sin \xi X] \qquad (\xi \in \mathbb{R}).$$

Then we have

$$\frac{V(\xi)}{\xi} = \int_0^{\infty -} D(x)\cos \xi x \, dx \qquad (\xi > 0).$$

Therefore, by Theorems 1.1 and 1.2, we obtain the following theorem:

Theorem 4.1. *Let $l \in R_0$. Assume that D is eventually non-increasing on $[0, \infty)$. Then*

(1) *$D(x) \sim x^{-1}l(x)$ as $x \to \infty$ if and only if $xV(1/x) \in \Pi_l$ in x with l-index* 1.

(2) *If*

$$V(\xi) \sim l(1/\xi)\frac{\pi}{2} \qquad (\xi \to 0+), \tag{4.1}$$

then

$$D \in \Pi_l \text{ with } l\text{-index } -1. \tag{4.2}$$

Conversely, if $\xi^{-1}V(\xi)$ is non-negative and non-increasing in ξ in a neighborhood of 0, *then* (4.2) *implies* (4.1).

Remark. If X is non-negative, then $D(x) = P(X > x)$ for $x > 0$, and so D is non-increasing on $(0, \infty)$.

5. Application to Stationary Processes

In this section, we apply Theorems 1.1 and 1.2 to stationary processes. Let $X = (X(t) : t \in \mathbb{R})$ be a real, weakly stationary process with zero expectation, and let R be the correlation function of $X : R(t) = E(X(t)X(0))$ for $t \in \mathbb{R}$. Let μ_x be the spectral measure of $X : R(t) = \int_{-\infty}^{\infty} e^{-it\xi}\mu_X(d\xi)$ for $t \in \mathbb{R}$. If μ_X is absolutely continuous with respect to the Lebesgue measure $d\xi$, then we call the density Δ the spectral density of $X : \mu_X(\xi) = \Delta(\xi)\, d\xi$. The spectral density Δ is a non-negative, even, and integrable function on $\mathbb{R}$.

Proposition 5.1. *Assume that the correlation function R is eventually monotone on $[0, \infty)$, $\lim_{t\to\infty} R(t) = 0$. Then the spectral measure of X is absolutely continuous with respect to the Lebesgue measure, and the spectral density Δ is given by*

$$\Delta(\xi) = \frac{1}{\pi}\int_0^{\infty-} R(t)\cos t\xi\, dt \qquad (t \in \mathbb{R}). \tag{5.1}$$

Proof. By the assumption, R is even, continuous, and either eventually non-negative and non-increasing, or eventually non-positive and non-decreasing. By Lévy's inversion formula, if a and b are continuity points of μ_X such that $0 < a < b$, then

$$\mu_X(a, b) = \lim_{M\to\infty} \frac{1}{2\pi}\int_{-M}^{M} \frac{e^{itb} - e^{ita}}{it} R(t)\, dt = F(b) - F(a),$$

where

$$F(\xi) := \frac{1}{\pi}\int_0^{\infty-} \frac{R(t)}{t}\sin \xi t\, dt \qquad (\xi > 0).$$

Since the improper integral $\int_0^{\infty-} R(t)\cos \xi t\, dt$ converges uniformly in $\xi > \varepsilon$ for any $\varepsilon > 0$, the function F is of C^1-class in $(0, \infty)$ and satisfies

$$F'(\xi) = \frac{1}{\pi}\int_0^{\infty-} R(t)\cos \xi t\, dt \qquad (\xi > 0).$$

Therefore μ_X is absolutely continuous in $(0, \infty)$, and the density there is equal to the derivative F'. If we put $\Delta(\xi) := F'(\xi)$ for $\xi > 0$, then we obtain

$$R(t) = \mu_X\{0\} + 2\int_0^{\infty} \Delta(\xi)\cos \xi t\, d\xi \qquad (t \in \mathbb{R}).$$

By the Riemann–Lebesgue Lemma, the second term on the right converges to zero as $t \to \infty$, so that $\mu_X\{0\} = 0$. This completes the proof. ∎

By Proposition 5.1, and Theorems 1.1 and 1.2, we immediately obtain the following theorem:

THEOREM 5.2. *Let $l \in R_0$. Assume that the correlation function R is eventually non-increasing on* $[0, \infty)$, $\lim_{t\to\infty} R(t) = 0$. *Then*

(1) *$R(t) \sim t^{-1}l(t)$ as $t \to \infty$ if and only if $\Delta(1/\cdot) \in \Pi_l$ with l-index π^{-1}.*

(2) *If*

$$\Delta(\xi) \sim \xi^{-1} l(1/\xi)\frac{1}{2} \qquad (\xi \to 0+), \tag{5.2}$$

then

$$R \in \Pi_l \qquad \text{with } l\text{-index } -1. \tag{5.3}$$

Conversely, if Δ is non-increasing in a neighborhood of 0, *then* (5.3) *implies* (5.2).

REFERENCES

1. N. H. BINGHAM AND J. L. TEUGELS, Mercerian and Tauberian theorems for differences, *Math. Z.* **170** (1980), 247–262.
2. N. H. BINGHAM, C. M. GOLDIE, AND J. L. TEUGELS, "Regular Variation," Cambridge Univ. Press, Cambridge, UK, 1987.

3. R. P. BOAS, "Integrability Theorems for Trigonometric Transforms," Springer-Verlag, Berlin, 1967.
4. L. DE HAAN, An Abel–Tauber theorem for Laplace transforms, *J. London Math. Soc. (2)* **13** (1976), 537–542.
5. E. J. G. PITMAN, On the behaviour of the characteristic function of a probability distribution in the neighbourhood of the origin, *J. Austral. Math. Soc.* **8** (1968), 422–443.
6. K. SONI AND R. P. SONI, Slowly varying functions and asymptotic behaviour of a class of integral transforms, II, *J. Math. Anal. Appl.* **49** (1975), 477–495.
7. E. C. TITCHMARSH, "Introduction to the Theory of Fourier Integrals," 2nd ed., Oxford Univ. Press, Oxford, 1948.